



Category: Physics

Type of Paper: Original Research Article

Received: July 23, 2026, **Revised:** September 3, 2026, **Accepted:** September 22, 2026

Published: October 9, 2026

DOI: [10.54503/0321-1339-2026.126.2-10](https://doi.org/10.54503/0321-1339-2026.126.2-10)

Continuum description of the edge states in a $(\times \mathbb{Z}_3)^3$ -symmetry-protected topological model

Mkhitar Mirumyan¹, Shahane A. Khachatryan¹, Ara G. Sedrakyan^{1*}

¹Alikhanyan National Science Laboratory (Yerevan Physics Institute), Alikhanian Br. str. 2, Yerevan 36, Armenia

*Correspondence: sedrak@mail.yerphi.am

Abstract

We investigate the continuum limit of the edge-state Hamiltonian of a two-dimensional symmetry-protected topological (SPT) model with $(\times \mathbb{Z}_3)^3 = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry. Starting from the lattice Hamiltonian, we first reformulate the model in terms of S_3 generators and subsequently introduce a constrained Schwinger-like fermionic representation. Within this formulation, the relevant operators are expressed as fermionic bilinears, while non-local Jordan–Wigner string operators do not appear explicitly. At the critical point, $\lambda = \bar{\lambda}$, we show that the continuum Hamiltonian supports gapless excitations with a linear dispersion relation, consistent with a conformal description of the edge theory. Using a mean-field treatment of the interaction terms, we further obtain a preliminary estimate of the finite-size correction to the ground-state energy. The corresponding effective central charge, $c = \sqrt{3} \simeq 1.73$, is remarkably close to the numerical value previously obtained for the $(\times \mathbb{Z}_3)^3$ SPT edge model and to the value predicted by the $SU(3)_2/SU(2)_2$ coset conformal field theory. Although this derivation remains heuristic, our results suggest that the constrained fermionic formulation provides a natural framework for describing the low-energy continuum limit of $(\mathbb{Z}_3)^3$ SPT edge states and may serve as a useful starting point for a more systematic derivation of the underlying conformal field theory.

Keywords: statistical physics, symmetry-protected topological phases, $\mathbb{Z}_3$ symmetry, chain Hamiltonian of edge states

1. Introduction

Symmetry-protected topological (SPT) phases are quantum phases of matter that lie beyond the conventional Landau paradigm of symmetry breaking [1, 2, 3]. Unlike conventionally ordered phases characterized by local order parameters, SPT phases possess a gapped bulk with short-range entanglement while supporting robust boundary excitations protected by global symmetries. As long as the protecting symmetry is preserved, these boundary states cannot be removed by local perturbations without either closing the bulk energy gap or breaking the symmetry. This bulk-boundary correspondence is one of the defining features of SPT phases [1, 4].

Two-dimensional SPT phases are particularly interesting because their one-dimensional boundaries provide a natural setting for realizing anomalous quantum systems. The symmetry action



at the boundary generally cannot be implemented in a purely one-dimensional lattice model as an onsite symmetry independently of the nontrivial bulk. This obstruction is characterized by a nontrivial 't Hooft anomaly, which implies that a symmetry-preserving boundary cannot, in general, be simultaneously gapped and nondegenerate. Consequently, the boundary must either remain gapless, spontaneously break the protecting symmetry, or, when allowed by dimensionality and the corresponding degrees of freedom, realize a more exotic boundary phase [5, 6].

Among discrete symmetry groups, the cyclic group $\mathbb{Z}_3$ provides the simplest extension of the familiar Ising $\mathbb{Z}_2$ symmetry while already exhibiting considerably richer physics. Bosonic SPT phases protected by $(\times \mathbb{Z}_3)^3$ symmetry are classified, within the group-cohomology construction, by

$$H^3((\times \mathbb{Z}_3)^3, U(1)) = (\times \mathbb{Z}_3)^7, \quad (1.1)$$

and therefore admit a large family of distinct SPT phases characterized by different cohomology classes [1, 4]. Their edge theories may carry the corresponding $(\times \mathbb{Z}_3)^3$ anomaly and therefore cannot, in general, be realized as ordinary one-dimensional systems with a purely onsite implementation of the same symmetry [6].

A convenient microscopic realization of such boundary theories is provided by quantum three-state Potts chains with appropriately chosen interactions. Depending on the Hamiltonian, the edge may exhibit stable gapless behavior described at low energies by conformal field theories (CFTs), whose operator content encodes the anomalous realization of the protecting symmetry. The critical properties of these edge states are characterized by universal quantities such as the central charge, scaling dimensions, and symmetry quantum numbers of the primary fields [7, 8].

Recent studies suggest that the edge states of $\mathbb{Z}_3$ -protected SPT phases can be naturally related to coset conformal field theories of the form $\frac{SU(3)_k}{SU(2)_k}$, with central charge $c = \frac{8k}{k+3} - \frac{3k}{k+2}$. For $k = 1$, this gives $c = 1$, in agreement with numerical studies of the simplest $\mathbb{Z}_3$ SPT edge Hamiltonian. More elaborate SPT constructions with enlarged symmetry, such as $(\mathbb{Z}_3)^3$, are conjectured to correspond to higher-level coset theories. In particular, the $k = 2$ theory gives $c = \frac{26}{15} \approx 1.73$, which is close to the numerical value obtained for the corresponding $(\mathbb{Z}_3)^{\otimes 3}$ SPT edge model. This suggests a possible unified conformal description of the critical edge dynamics of this class of models [8, 9, 10, 11, 12, 13].

An important aspect of these edge theories is the realization of the protecting $(\times \mathbb{Z}_3)^3$ symmetry. Rather than acting trivially on the low-energy degrees of freedom, the symmetry should be encoded in the current algebra of the continuum theory and in the transformation properties of its primary fields. Demonstrating that the lattice edge model and the candidate continuum CFT realize the same $\mathbb{Z}_3$ anomaly would provide strong evidence that they belong to the same universality class [6]. Understanding this correspondence remains an important problem in the theory of SPT phases and illustrates the deep relation between topology, symmetry, and conformal field theory [2, 3].

In the present work, we consider the edge Hamiltonian of a two-dimensional paramagnetic SPT Potts model with $(\times \mathbb{Z}_3)^3$ symmetry and investigate its continuum limit in terms of fermionic fields. Starting from the lattice Hamiltonian, we reformulate the relevant operators using a constrained Schwinger-like fermionic representation and derive the corresponding low-energy continuum Hamiltonian. Our aim is to clarify the origin of the gapless edge excitations and to establish a direct connection between the microscopic lattice description and the proposed $SU(3)_2/SU(2)_2$ coset conformal field theory.

2. Continuum limit of the edge Hamiltonian

The edge Hamiltonian of the discrete $(\times \mathbb{Z}_3)^3 = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ chain model was introduced in [11]. Let us write it with arbitrary real couplings $\lambda, \bar{\lambda}$ in the following form:

$$H(\lambda, \bar{\lambda}) = -\lambda \sum_i (X_i + X_i^+) - \bar{\lambda} \sum_i (Z_{i-1}^+ (X_i + X_i^+) Z_{i+1} + H. c.) \quad (2.2)$$

Here Z_i and X_i are the usual $\mathbb{Z}_3$ symmetric matrix generators attached to each site i of the discrete spin-1 ($|v_1\rangle_i, |v_2\rangle_i, |v_3\rangle_i$) vector chain:

$$Z_i^3 = 1, \quad X_i^3 = 1, \quad Z_i X_i = \varepsilon X_i Z_i, \quad Z_i |v_k\rangle_i = \varepsilon^k |v_k\rangle_i, \quad X_i |v_k\rangle_i = |v_{k+1}\rangle_i, \quad \varepsilon^3 = 1. \quad (2.3)$$

A direct attempt to take the continuum limit of this lattice Hamiltonian by setting

$$Z_i = Z(x_i), \quad x_{i\pm 1} = x_i \pm a, \quad a \rightarrow 0, \quad \text{and} \quad Z_{i\pm 1} = Z_i \pm a \partial Z_i, \quad (2.4)$$

leads to the following expression.

$$\begin{aligned} H(\lambda, \bar{\lambda}) &= -\lambda \sum_i (X_i + X_i^+) - \bar{\lambda} \sum_i [(Z_i^+ - a \partial Z_i^+) (X_i + X_i^+) (Z_i + a \partial Z_i) + h. c.] \\ &= -(\lambda - \bar{\lambda}) \sum_i (X_i + X_i^+) - \bar{\lambda} a (\varepsilon - \varepsilon^+) \sum_i [Z_i^+ \partial Z_i (X_i - X_i^+) + (X_i - X_i^+) Z_i^+ \partial Z_i]. \end{aligned} \quad (2.5)$$

Here a is the lattice spacing, and we have used the relations

$$\varepsilon + \varepsilon^+ = -1 \quad \text{and} \quad Z_i^+ \partial Z_i = -\partial Z_i^+ Z_i \quad (2.6)$$

The operator Z can be expressed in terms of the Hermitian operator S_z as $Z = \varepsilon^{S_z}$, where $\varepsilon = e^{i2\pi/3}$. This relation implies

$$Z^+ \partial Z = \ln \varepsilon \partial S_z, \quad Z \partial Z^+ = -\ln \varepsilon \partial S_z. \quad (2.7)$$

These relations can be treated as a kind of bosonization of the $\mathbb{Z}_3$ degrees of freedom. We observe that the last term in (2.5) is anti-Hermitian, although the original lattice Hamiltonian is Hermitian. This indicates that some care is required in performing the continuum expansion and, in particular, in treating the Hermitian-conjugate contribution.

Let us write the lattice expression (2.2) first in the form

$$H(\lambda, \bar{\lambda}) = -\lambda \sum_i X_i - \bar{\lambda} \sum_i [Z_{i-1}^+ X_i Z_{i+1} + Z_{i-1} X_i Z_{i+1}^+] + [h. c.]. \quad (2.8)$$

To obtain the continuum limit of the Hamiltonian (2.8) we use Eqs. (2.4) and (2.6) and replace $\sum_i \rightarrow \int$. The Hamiltonian becomes

$$H(\lambda, \bar{\lambda}) = -\lambda \int X - \bar{\lambda} (\varepsilon + \varepsilon^+) \int X - a \bar{\lambda} (\varepsilon - \varepsilon^+) \int (Z^+ \partial Z X + X Z^+ \partial Z) + [h. c.]. \quad (2.9)$$

Now, writing explicitly the Hermitian conjugate part one obtains the Hermitian expression

$$H(\lambda, \bar{\lambda}) = -(\lambda - \bar{\lambda}) \int (X + X^+) - \bar{\lambda} a (\varepsilon - \varepsilon^+) \int \{Z^+ \partial Z, (X + X^+)\}. \quad (2.10)$$

Using (2.7), we can represent the Hamiltonian as follows

$$H(\lambda, \bar{\lambda}) = -(\lambda - \bar{\lambda}) \int (X + X^+) - \bar{\lambda} a \frac{2\pi}{\sqrt{3}} \int \{X + X^+, \partial S_z\}. \quad (2.11)$$

It is useful to make a few remarks concerning this continuum expression:

Note 1. In the basis where the operator X is diagonal, the first term in (2.11), when rewritten in the particle representation, should contain a mass term. Consequently, at $\lambda = \bar{\lambda}$ the corresponding particle excitations become massless. This is consistent with the conformal field theory prediction of [11].

Note 2. For completeness, we can rewrite the operator X in terms of the new operator S_ϕ , defined through $X = \varepsilon^{S_\phi}$,

$$S_\phi = \frac{i}{\sqrt{3}} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}. \quad (2.12)$$

This operator can also be expressed in terms of the S_3 generators $X^\pm$ as $S_\phi = \frac{i}{\sqrt{3}} (X^+ - X)$. Replacing the combination $(X + X^+)$ by $2S_0 + 2(\cos[2\pi/3] - 1)S_\phi^2$, where S_0 denotes the identity operator, the Hamiltonian takes the form

$$H(\lambda, \bar{\lambda}) = (\bar{\lambda} - \lambda) \int (2S_0 - 3S_\phi^2) - \bar{\lambda} a \frac{2\pi}{\sqrt{3}} \int \{2S_0 - 3S_\phi^2, \partial S_z\}, \quad (2.13)$$

Note 3. It is also possible to express the operators S_0, S_ϕ, S_z in terms of the generators of the $SU(3)$ group:

$$S_\phi = \frac{1}{\sqrt{3}}(\lambda_2 - \lambda_5 + \lambda_7),$$

$$S_z = \frac{1}{2}(\lambda_3 + \sqrt{3}\lambda_8), \quad X + X^+ = \lambda_1 + \lambda_4 + \lambda_6,$$

where $\lambda_i, i = 1, 2, \dots, 8$ are the Gell-Mann matrices.

Fermionization.

The Jordan-Wigner transformation relates a spin-1/2 chain to a system of spinless fermions [14], which is particularly useful for the diagonalization of models whose fermionic Hamiltonians are quadratic, such as the Ising and XY models [15, 16]. For symmetry-protected topological spin models, an alternative description is usually formulated in terms of parafermions [8, 17]. Here, however, we present a formulation in terms of ordinary fermions [16, 18, 19].

A direct higher-dimensional generalization of the Jordan–Wigner transformation to spin-1 models, formulated in terms of two species of spinless fermions, generally produces a Hamiltonian containing non-local terms [16, 20, 21]. Attaching two kinds of fermions $c_\uparrow, c_\downarrow$ to each lattice site, the $\mathbb{Z}_3$ generators can be represented as follows (below $n_{\uparrow/\downarrow} = c_{\uparrow/\downarrow}^\dagger c_{\uparrow/\downarrow}$).

$$X = (1 - n_\uparrow)c_\downarrow^\dagger + (1 - n_\downarrow)c_\uparrow + c_\uparrow^\dagger c_\downarrow, \quad X^+ = (1 - n_\uparrow)c_\downarrow + (1 - n_\downarrow)c_\uparrow^\dagger + c_\downarrow^\dagger c_\uparrow,$$

$$Z = 1 + 2n_\uparrow n_\downarrow + (\varepsilon - 1)n_\uparrow + (\varepsilon^* - 1)n_\downarrow, \quad Z^+ = 1 + 2n_\uparrow n_\downarrow + (\varepsilon^* - 1)n_\uparrow + (\varepsilon - 1)n_\downarrow. \quad (2.14)$$

To take into account the anticommutation relations of fermionic operators attached to different lattice sites, one must introduce a generalized Jordan–Wigner transformation by equipping each fermionic operator with a product of non-local string operators,

$$c_{\alpha,i}^\pm \rightarrow c_{\alpha,i}^\pm e^{\mp i\pi \sum_{k<i, \alpha=\uparrow, \downarrow} c_{\alpha,k}^\dagger c_{\alpha,k}} \quad (2.15)$$

The odd powers of the particle creation and annihilation operators appearing in $X^\pm$ then generate non-local fermionic terms in the Hamiltonian (2.2).

Thus, in order to avoid the problem of non-locality, we introduce a set of three fermionic operators $c_\alpha, c_\alpha^\dagger, \alpha = 1, 2, 3$, attached to each site of the chain. We interpret the species index $\alpha = 1, 2, 3$ as a color degree of freedom. The three basis states are identified as $c_\alpha^\dagger |0\rangle = |\alpha\rangle, \alpha = 1, 2, 3$. The $\mathbb{Z}_3$ operators Z and X are chosen in the form

$$Z = n_1 + n_2 n_3 + \varepsilon(n_2 + n_1 n_3) + \varepsilon^+(n_3 + n_1 n_2) = \sum_{\alpha=1}^3 \varepsilon^{\alpha-1} (n_\alpha + n_{\alpha+1} n_{\alpha+2}),$$

$$X = c_1^\dagger c_2 (1 - n_3) + c_2^\dagger c_3 (1 - n_1) + c_3^\dagger c_1 (1 - n_2) = \sum_{\alpha=1}^3 c_\alpha^\dagger c_{\alpha+1} (1 - n_{\alpha+2}), \quad (2.16)$$

$$n_\alpha = c_\alpha^\dagger c_\alpha, \quad ZX = \varepsilon XZ, \quad c_{\alpha+3} = c_\alpha, \quad \varepsilon^3 = 1.$$

These operators annihilate the unphysical states $|0\rangle, |\alpha, \beta\rangle_{\alpha \neq \beta}$, and $|1, 2, 3\rangle$. The physical Hilbert space is therefore restricted to the single-occupancy sector, $\sum_\alpha n_\alpha = 1$, and the corresponding projector can be written as $P = \delta(\sum_{\alpha=1}^3 n_\alpha - 1)$.

In this “Schwinger-type” fermionization, the Jordan–Wigner non-local string operator $e^{i\pi \sum_{k,\alpha}^{N,3} n_{k\alpha}}$ does not appear explicitly in the Hamiltonian, since all $\mathbb{Z}_3$ generators are represented by expressions containing even powers of the fermionic creation and annihilation operators.

Note 4. For the full lattice, the total projector $\prod_k \delta(\sum_{\alpha=1}^3 n_{\alpha k} - 1)$ appearing in the transfer matrix $\text{Tr}[\exp(-H/T)]$ can be represented as an integral over auxiliary scalar fields J_k :

$$\prod_k \delta(\sum_{\alpha=1}^3 n_{\alpha k} - 1) = \int \prod_k dJ_k \exp\{iJ_k(\sum_{\alpha=1}^3 n_{\alpha k} - 1)\} = \int \prod_k dJ_k \exp iJ_k (\text{Tr} S_k^0 - 1). \quad (2.17)$$

Here Tr denotes the trace of the matrix S_k^0 over the color indices. By combining (2.13) with (2.17), one obtains the following continuum limit of the Hamiltonian (2.8)

$$H(\lambda, \bar{\lambda}, J(x), c_\alpha(x)) = (\bar{\lambda} - \lambda) \int (2S_0 - 3S_\phi^2) - \frac{2\pi a \bar{\lambda}}{\sqrt{3}} \int \{2S_0 - 3S_\phi^2, \partial S_z\} - i \int J(x) (Tr S^0 - 1), \quad (2.18-2.19)$$

where the operators

$$S_\phi = \frac{i}{\sqrt{3}} (X^+ - X) \quad \text{and} \quad S_z = \frac{i}{\sqrt{3}} (Z^+ - Z) \quad (2.20)$$

can be represented entirely by colored fermionic operators $c_\alpha(x)$, $c_\alpha^+(x)$, defined by (2.16), whereas Lagrange multipliers TJ_k form a scalar field $J(x)$.

3. Ground state energy calculations: fermionic expressions and some manipulations

Let us rewrite the expression (2.19) at the critical point $\lambda = \bar{\lambda}$ in an explicit fermionic representation, employing a mean-field approximation and imposing periodic boundary conditions. Our main goal is to estimate the ground-state energy within this approximation. At this stage, we do not consider the Hamiltonian supplemented by the auxiliary scalar field $J(x)$. Instead, we evaluate the Hamiltonian (2.8) directly using the fermionic representation (2.16), restricted to the three-dimensional physical Hilbert space spanned by single-state basis states $|\alpha\rangle$, $\alpha = 1, 2, 3$. All operators that vanish on these vectors must be ignored.

Taking into account the restriction to the physical fermionic Hilbert space imposed by the projectors, at $\lambda = \bar{\lambda}$ the surviving term of the Hamiltonian can be expressed in terms of the fermionic operators as follows (below we use the notation $n_\alpha(x) = c_\alpha^+(x)c_\alpha(x)$):

$$-\bar{\lambda} \int_0^L dx \sum_\alpha [c_\alpha^+(x)c_{\alpha+1}(x) \partial_x n(x)_{\alpha+1} - \partial_x n(x)_\alpha c_\alpha^+(x)c_{\alpha+1}(x)] + h.c. \quad (3.21)$$

This Hamiltonian can be separated into two parts: a quadratic contribution and a fourth-order fermionic interaction term,

$$\begin{aligned} & \bar{\lambda} \int_0^L dx \sum_\alpha [(\partial_x c_\alpha^+(x))c_{\alpha+1}(x) - c_\alpha^+(x)(\partial_x c_{\alpha+1}(x)) + h.c.] \\ & + \bar{\lambda} \int_0^L dx \sum_\alpha [c_\alpha^+(x)(\partial_x c_{\alpha+1}(x))n(x)_{\alpha+1} - n(x)_\alpha(\partial_x c_\alpha^+(x))c_{\alpha+1}(x) + h.c.] \end{aligned} \quad (3.22)$$

The first term can be evaluated directly in the Fourier basis, i.e., in terms of fermions in momentum space. As usual, we introduce the continuum normalization $c(x) = \frac{c_i}{\sqrt{a}}$, $\int_0^L dx = a \sum_i^N$, and $c(p) = \frac{1}{L} \int_0^L dx c(x) e^{-2\pi i \frac{px}{L}}$.

For each momentum p , one is then led to diagonalize the following Hermitian operator in the space of single-particle states generated by the creation operators $c_\alpha^+(p)$ acting on the vacuum state $|0\rangle$:

$$ip \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}. \quad (3.23)$$

The eigenvalues of this operator, $\pm\sqrt{3}p, 0$, indicate that, within this first approximation, the ground state is constructed by occupying the single-particle states corresponding to the negative-energy branch.

$$|v_0(p)\rangle = \frac{1}{2\sqrt{3}} \{(i\sqrt{3} - 1)c_1^+(p) - (i\sqrt{3} + 1)c_2^+(p) + 2c_3^+(p)\} |0\rangle,$$

whose energy is $\sim -\int dp \sqrt{3}p$.

For the second term, as a preliminary mean-field approximation, we replace the particle-number operators by their expectation values in the approximate ground state,

$$\langle v_0(p) | n_\alpha(p) | v_0(p) \rangle = \frac{1}{3}, \quad (3.24)$$



which is consistent with the requirement $\sum_{\alpha=1}^3 n_{\alpha} = 1$ and then evaluate the remaining quadratic fermionic Hamiltonian in the standard way. The condition $\langle |n_{\alpha}(p)| \rangle = \frac{1}{3}$ was also set to ensure Z_3 invariance of the ground state in color space $\alpha = 1, 2, 3$, and we have seen that the chosen ground state preserves this symmetry (3.24). This replacement produces an additional contribution to the ground-state energy of the form $\sim \frac{1}{L} \int dp \frac{p}{\sqrt{3}}$. The corresponding numerical coefficients should, of course, be corrected within a more refined treatment of the interaction term.

Note 5. Another interesting feature of this heuristic calculation is that, at the critical point $\lambda = \bar{\lambda}$, assuming that the ground-state energy of the second term can be approximated as $\frac{\pi\sqrt{3}}{6L}$, one obtains the central charge $c = \sqrt{3} \approx 1.73$ by comparing this finite-size correction with the universal Casimir contribution to the ground-state energy for periodic geometries [7, 22, 23]. This value is remarkably close to the central charge expected for the proposed continuum description, although the present estimate should be regarded as preliminary because it relies on the mean-field treatment of the interaction term.

For the Z_{2N} -symmetric model with a similar Hamiltonian, analogous calculations, in which the spin degrees of freedom are represented by $2N$ species of spinless fermions, lead to higher-order derivative terms in the continuum limit.

4. Conclusion

In this work, we have derived the continuum limit of the edge-state Hamiltonian for a $\mathbb{Z}_3$ symmetry-protected topological model. The lattice Hamiltonian was first reformulated in terms of S_3 generators and subsequently represented using a constrained fermionic (Schwinger-like) formulation. An attractive feature of this representation is that the relevant operators are expressed entirely in terms of fermionic bilinears, while the non-local Jordan–Wigner string operators do not appear explicitly.

At the critical point $\lambda = \bar{\lambda}$, the continuum Hamiltonian naturally separates into a kinetic contribution and an interaction term. The kinetic part gives rise to gapless excitations with a linear dispersion relation, in agreement with the expected conformal description of the edge theory. Within a simple mean-field treatment of the interaction term, we obtained a preliminary estimate of the finite-size correction to the ground-state energy. Remarkably, this heuristic calculation leads to the value

$$c = \sqrt{3} \approx 1.73,$$

which is very close to the numerical result for the central charge reported in Ref. [11] and supports the proposed conformal description of the critical edge theory.

Although the present derivation relies on several approximations, the results suggest that the constrained fermionic formulation provides a natural framework for describing the low-energy continuum theory of $\mathbb{Z}_3$ SPT edge states. A more systematic treatment of the interaction term, together with an explicit implementation of the projector enforcing the physical Hilbert space, should allow for a controlled derivation of the effective conformal field theory and its finite-size spectrum. It would also be interesting to extend this approach to higher $\mathbb{Z}_N$ and non-Abelian symmetry-protected topological phases, for which analogous fermionic continuum descriptions may emerge.

References

- [1] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Symmetry protected topological orders and the group cohomology of their symmetry group, Phys. Rev. B 87, 155114 (2013).
<https://doi.org/10.1103/PhysRevB.87.155114>



- [2] T. Senthil, Symmetry-protected topological phases of quantum matter, *Annu. Rev. Condens. Matter Phys.* 6, 299–324 (2015). <https://doi.org/10.1146/annurev-conmatphys-031214-014740>
- [3] X.-G. Wen, Colloquium: Zoo of quantum-topological phases of matter, *Rev. Mod. Phys.* 89, 041004 (2017). <https://doi.org/10.1103/RevModPhys.89.041004>
- [4] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Symmetry-protected topological orders in interacting bosonic systems, *Science* 338, 1604–1606 (2012). <https://doi.org/10.1126/science.1227224>
- [5] Y.-M. Lu and A. Vishwanath, Theory and classification of interacting integer topological phases in two dimensions: A Chern-Simons approach, *Phys. Rev. B* 86, 125119 (2012). <https://doi.org/10.1103/PhysRevB.86.125119>
- [6] D. V. Else and C. Nayak, Classifying symmetry-protected topological phases through the anomalous action of the symmetry on the edge, *Phys. Rev. B* 90, 235137 (2014). <https://doi.org/10.1103/PhysRevB.90.235137>
- [7] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory* (Springer, New York, 1997). <https://doi.org/10.1007/978-1-4612-2256-9>
- [8] V. A. Fateev and A. B. Zamolodchikov, Parafermionic currents in the two-dimensional conformal quantum field theory and selfdual critical points in $Z(n)$ invariant statistical systems, *Sov. Phys. JETP* 62, 215–225 (1985).
- [9] P. Goddard, A. Kent, and D. Olive, Virasoro algebras and coset space models, *Phys. Lett. B* 152, 88–92 (1985). [https://doi.org/10.1016/0370-2693\(85\)91145-1](https://doi.org/10.1016/0370-2693(85)91145-1)
- [10] P. Goddard, A. Kent, and D. Olive, Unitary representations of the Virasoro and super-Virasoro algebras, *Commun. Math. Phys.* 103, 105–119 (1986). <https://doi.org/10.1007/BF01464283>
- [11] H. Topchyan, V. Iugov, M. Mirumyan, Sh. Khachatryan, T. Hakobyan, and T. Sedrakyan, Z_3 and $(\times Z_3)^3$ symmetry protected topological paramagnets, *J. High Energy Phys.* 2023, 199 (2023). [https://doi.org/10.1007/JHEP12\(2023\)199](https://doi.org/10.1007/JHEP12(2023)199)
- [12] H. Topchyan, V. Iugov, M. Mirumyan, T. S. Hakobyan, T. A. Sedrakyan, and A. G. Sedrakyan, Two-dimensional topological paramagnets protected by Z_3 symmetry: Properties of the boundary Hamiltonian, *SciPost Phys.* 18, 068 (2025). <https://doi.org/10.21468/SciPostPhys.18.2.068>
- [13] H. Topchyan, T. Hakobyan, M. Mirumyan, T. A. Sedrakyan, and A. Sedrakyan, Topological edge states in two-dimensional Z_4 Potts paramagnet protected by the $Z_4^{\times 3}$ symmetry, *Phys. Rev. B* 114, 065111 (2026). <https://doi.org/10.1103/xszt-gsss>
- [14] P. Jordan and E. Wigner, Über das Paulische Äquivalenzverbot, *Z. Phys.* 47, 631–651 (1928). <https://doi.org/10.1007/BF01331938>
- [15] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982).
- [16] Sh. Khachatryan and A. Sedrakyan, Characteristics of two-dimensional lattice models from a fermionic realization: Ising and XYZ models, *Phys. Rev. B* 80, 125128 (2009). <https://doi.org/10.1103/PhysRevB.80.125128>
- [17] M. Levin and Z.-C. Gu, Braiding statistics approach to symmetry-protected topological phases, *Phys. Rev. B* 86, 115109 (2012). <https://doi.org/10.1103/PhysRevB.86.115109>
- [18] A. Avakyan, T. Hakobyan, and A. Sedrakyan, A family of affine quantum group invariant integrable extensions of the Hubbard Hamiltonian, *Nucl. Phys. B* 490, 633–652 (1997). [https://doi.org/10.1016/S0550-3213\(97\)00067-9](https://doi.org/10.1016/S0550-3213(97)00067-9)
- [19] J. Ambjørn, Sh. Khachatryan, and A. Sedrakyan, Simplified tetrahedron equations: Fermionic realization, *Nucl. Phys. B* 734, 287–303 (2006). <https://doi.org/10.1016/j.nuclphysb.2005.11.023>
- [20] F. Göhmann and S. Murakami, Fermionic representations of integrable lattice systems, *J. Phys. A: Math. Gen.* 31, 7729–7749 (1998). <https://doi.org/10.1088/0305-4470/31/38/009>
- [21] J. Ambjørn, D. Karakhanyan, M. Mirumyan, and A. Sedrakyan, Fermionization of the spin-S Uimin–Lai–Sutherland model: Generalisation of the supersymmetric t–J model to spin-S, *Nucl. Phys. B* 599, 547–560 (2001). [https://doi.org/10.1016/S0550-3213\(00\)00757-4](https://doi.org/10.1016/S0550-3213(00)00757-4)



- [22] J. L. Cardy, Conformal invariance and universality in finite-size scaling, J. Phys. A: Math. Gen. 17, L385–L387 (1984). <https://doi.org/10.1088/0305-4470/17/7/003>
- [23] I. Affleck, Universal term in the free energy at a critical point and the conformal anomaly, Phys. Rev. Lett. 56, 746–748 (1986). <https://doi.org/10.1103/PhysRevLett.56.746>

Author Contributions Statement (CRediT): Mkhitar Mirumyan: Investigation, Formal analysis; Shahane A. Khachatryan: Investigation, Writing—original draft; Ara G. Sedrakyan: Investigation, Writing—review & editing.

Conflicts of Interest: The authors declare no conflicts of interest.

Funding: The authors acknowledge support from the Higher Education and Science Committee of MESCS RA grants 21AG-1C024 (Sh.Kh., A.S., M.M.) and 24FP-1F039 (Sh.Kh., A.S.).

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of REPNAS and/or the editor(s). REPNAS and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.